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$(Z_2)^k$-ACTIONS WHOSE FIXED DATA HAS A SECTION

PEDRO L. Q. PERGHER

ABSTRACT. Given a collection of $2^k - 1$ real vector bundles ε_a over a closed manifold F, suppose that, for some a_0, ε_{a_0} is of the form $\varepsilon'_{a_0} \oplus R$, where $R \to F$ is the trivial one-dimensional bundle. In this paper we prove that if $\bigoplus_a \varepsilon_a \to F$ is the fixed data of a $(Z_2)^k$-action, then the same is true for the Whitney sum obtained from $\bigoplus_a \varepsilon_a$ by replacing ε_{a_0} by ε'_{a_0}. This stability property is well-known for involutions. Together with techniques previously developed, this result is used to describe, up to bordism, all possible $(Z_2)^k$-actions fixing the disjoint union of an even projective space and a point.

1. INTRODUCTION

Consider $(Z_2)^k$ as the group generated by k commuting involutions $T_1, T_2, ..., T_k$. Being given a collection of $2^k - 1$ real vector bundles ε_a over a closed manifold F, where a runs through the nontrivial representations of $(Z_2)^k$, it is natural to ask whether the Whitney sum $\bigoplus_a \varepsilon_a \to F$ is the fixed data of some $(Z_2)^k$-action (M, Φ), $\Phi = (T_1, T_2, ..., T_k)$. The main aim of this paper is to prove the following fact related to this question: if $\bigoplus_a \varepsilon_a \to F$ is the fixed data of a $(Z_2)^k$-action and, for some a_0, ε_{a_0} is of the form $\varepsilon'_{a_0} \oplus R$, where $R \to F$ is the trivial one-dimensional bundle, then the same is true for the Whitney sum obtained from $\bigoplus_a \varepsilon_a$ by replacing ε_{a_0} by ε'_{a_0}. This extends for any k the well-known result of Conner and Floyd for $k = 1$ [3] and the extension for $k = 2$ obtained recently by Pergher [5]. The approach used in [5] to solve the case $k = 2$ was first to determine conditions in terms of Whitney numbers for $\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3 \to F$ to be the fixed data of a $(Z_2)^2$-action, by using essentially the "fixed point exact sequence of bordism of $((Z_2)^k, q)$-manifold-bundles", introduced by Stong in [10]; next we showed that if $(\varepsilon_1 \oplus R) \oplus \varepsilon_2 \oplus \varepsilon_3 \to F$ satisfies these conditions, then the same occurs with $\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3 \to F$. Although theoretically possible, in practice this trick does not work in the general case, because the complexity of the conditions on the Whitney numbers mentioned above increases considerably with k. Our approach will consist in using the fixed point sequence in a more direct way, without considerations about characteristic numbers step by step as in the case $k = 2$.

As we have seen in [5], the above result for $k = 2$ together with facts from [7] made it possible to obtain, up to bordism, all possible $(Z_2)^2$-actions fixing the disjoint union of an even projective space $RP(2n)$ and a point. In this paper we will complete this classification for any k, by putting together in a similar

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way the above general case and facts from [7]. Referring to this classification, we remark that in [7] we developed a method to analyse the bordism classes of $(Z_2)^k$-actions fixing the disjoint union of a connected closed n-dimensional manifold V^n and a point p, having as a starting point the knowledge of the set $\mathcal{A}$ of all equivariant bordism classes of involutions fixing $V^n \cup \{p\}$. This method has shown itself particularly effective when $\mathcal{A}$ has a single element, because in this case we have seen that if (M, Φ) is a $(Z_2)^k$-action fixing $V^n \cup \{p\}$, then its fixed data bears, in terms of bordism, a strong resemblance to the fixed data of $\sigma\Gamma_t^k(W,T)$, where $[(W,T)]$ is the only element of $\mathcal{A}$ and $\sigma\Gamma_t^k$ denotes certain operations which produce special $(Z_2)^k$-actions from a given involution. As an illustration of this effectiveness we have proved further that (M, Φ) is really bordant to some action of type $\sigma\Gamma_t^k(W,T)$ for $V^n = S^n$, $S^p \times S^q$ or $RP(2p+1)$ (see [7] and [8]). However, if $\mathcal{A}$ has more than one element the classification seems much more difficult, because in this case the results of [7] indicate the possibility of other classes added to those produced by the operations $\sigma\Gamma_t^k$; as we have seen in [5], this is exactly what happens when $V^n = RP(2r)$ (Royster has proved in [2] that in this case $\mathcal{A}$ has more than one element), and in this case the method of [7] is not enough to determine the classification. Our main theorem thus constitutes an additional tool to handle this question, and the above classification for any k is an interesting example illustrating both the need for our theorem and the generality of the results of [7].

2. Preliminaries

We begin with some general concepts, which are based on [10] and [7]. Being given a $(Z_2)^k$-action (M, Φ), $\Phi = (T_1, T_2, ..., T_k)$, with fixed point set F, the normal bundle η of F in M decomposes as a Whitney sum of subbundles on which $(Z_2)^k$ acts as one of the irreducible (nontrivial) real representations. This decomposition may be described by using sequences $a = (a_1, a_2, ..., a_k)$, where each a_j is either 0 or 1: if $\varepsilon_a \subset \eta$ denotes the subbundle on which each T_j acts as multiplication by $(-1)^{a_j}$ for each j, then

$$\eta = \bigoplus_{a \neq (0)} \varepsilon_a,$$

where $(0) = (0, 0, ..., 0)$ is the trivial sequence. First choosing an order for $\{a; a \neq (0)\}$, the fixed data of (M, Φ) will then be constituted by F and the ordered set of $2^k - 1$ vector bundles ε_a $(a \neq (0))$. Throughout this paper we will adopt the order given inductively by $(1, c_1), (1, c_2), ..., (1, c_{2^{k-1}-1}), (1, 0, 0, ..., 0), (0, c_1), (0, c_2), ..., (0, c_{2^{k-1}-1})$, where $c_1, c_2, ..., c_{2^{k-1}-1}$ denote the ordered irreducible nontrivial representations of $(Z_2)^{k-1}$ (the case $k = 1$ is trivial).

Given any automorphism $\sigma : (Z_2)^k \to (Z_2)^k$, one obtains a new $(Z_2)^k$-action from (M, Φ) by taking $(M; \sigma(T_1), \sigma(T_2), ..., \sigma(T_k))$; we denote this action by $\sigma(M, \Phi)$. The fixed data of $\sigma(M, \Phi)$ is obtained from the fixed data of (M, Φ) by a permutation of subbundles, obviously depending on σ. It is important to emphasize that not every configuration obtained by a permutation of the subbundles of the fixed data of (M, Φ) is derived from some automorphism, for the simple reason that in general the number of such configurations is greater than the number of bases of $(Z_2)^k$ (for $k = 2$ these numbers are equal). For example, for the $(Z_2)^k$-actions $\Gamma_t^k(W,T)$, $1 \leq t \leq k$, that will be defined in Section 4, the number of such configurations, which is

$$\frac{(2^k-1)!}{(2^{t-1})!(2^{t-1}-1)!(2^k-2^t)!},$$

is greater than the number of bases of $(Z_2)^k$ for $k \geq 5$ and $t \geq k-2$. If some configuration is not derived from an automorphism, we cannot guarantee in principle that it is the fixed data of a $(Z_2)^k$-action (this justifies the need for Lemma 1 of the next section for the general case $k \geq 2$).

According to [10] one has the bordism groups $\mathcal{N}_{m,n_1,\ldots,n_q}((Z_2)^k, q)$ constituted by the bordism classes of the so-called "$((Z_2)^k, q)$-manifold-bundles" $(M, \mu, \xi_1, \mu_1, \ldots, \xi_q, \mu_q)$, where M is a closed differentiable m-dimensional manifold, μ is a differentiable action of $(Z_2)^k$ on M, ξ_i is an n_i-dimensional real vector bundle over M, and μ_i is an action of $(Z_2)^k$ on the total space $E(\xi_i)$ by real vector bundle maps covering the action μ on M. $\widehat{\mathcal{N}}_{m,n_1,\ldots,n_q}((Z_2)^k, q)$ means the same group with the additional requirement that the first involution T_1 acts without fixed points.

Connecting these groups, one has the fixed point exact sequence of $((Z_2)^k, q)$-manifold-bundles

$$0 \to \mathcal{N}_{m,n_1,\ldots,n_q}((Z_2)^k, q) \xrightarrow{F} \bigoplus \mathcal{N}_{m',m'',n_1',n_1'',\ldots,n_q',n_q''}((Z_2)^{k-1}, 2q+1) \xrightarrow{S} \widehat{\mathcal{N}}_{m-1,n_1,\ldots,n_q}((Z_2)^k, q) \to 0,$$

where the sum is over all sequences with $m' + m'' = m$ and $n_i' + n_i'' = n_i$. For the description of F and S, see Proposition 3 on page 784 of [10]. The homomorphism S additionally maps the summand $\mathcal{N}_{m-1,1,n_1,0,\ldots,n_q,0}((Z_2)^{k-1}, 2q+1)$ isomorphically onto $\widehat{\mathcal{N}}_{m-1,n_1,\ldots,n_q}((Z_2)^k, q)$, and the inverse for S on this summand is also described in the proof of Proposition 3 of [10].

3. Proof of the main theorem

In order to prove our result we first need some lemmas.

Lemma 1. *Let (M, Φ), $\Phi = (T_1, T_2, \ldots, T_k)$, be a $(Z_2)^k$-action with fixed data $\bigoplus_a \varepsilon_a \to F$. Take arbitrary nontrivial representations $a = (a_1, a_2, \ldots, a_k)$, $b = (b_1, b_2, \ldots, b_k)$ of $(Z_2)^k$, and suppose $\varepsilon_a = \eta$. Then there is an automorphism $\sigma : (Z_2)^k \to (Z_2)^k$ such that the fixed data of $\sigma(M, \Phi)$ has $\varepsilon_b = \eta$.*

Proof. It suffices to consider $b = (1, 0, 0, \ldots, 0)$. Consider the homomorphism $f_a : (Z_2)^k \to Z_2$ given by $f_a(T_i) = (-1)^{a_i}$, and choose $\tau_2, \tau_3, \ldots, \tau_k$ generating $\ker(f_a)$ and $\tau_1 \notin \ker(f_a)$. Then the automorphism $\sigma : (Z_2)^k \to (Z_2)^k$ defined by $\sigma(T_i) = \tau_i$, $1 \leq i \leq k$, clearly works. ☐

The next lemma was inspired by the remark after Proposition 8 of [9, page 67].

Lemma 2. *Let*

$$F : \mathcal{N}_{m,n_1,\ldots,n_q}((Z_2)^k, q) \to \bigoplus \mathcal{N}_{m',m'',n_1',n_1'',\ldots,n_q',n_q''}((Z_2)^{k-1}, 2q+1)$$

be the monomorphism of the fixed point exact sequence, and suppose $F(\alpha) = \beta$ with $(N, \psi, \nu, \overline{\psi}, \eta_1, \psi_1, \eta_1', \psi_1', \ldots, \eta_q, \psi_q, \eta_q', \psi_q')$ being a representative for β. Then it is possible to describe an explicit representative for α in terms of N, ψ, ν, $\overline{\psi}$, η_i, ψ_i, η_i', ψ_i', $1 \leq i \leq q$.

Proof. Consider the $((Z_2)^k, q)$-manifold-bundle

$$(RP(\nu \oplus R), \mu, \eta_1 \oplus (\eta_1' \otimes \lambda), \mu_1, ..., \eta_q \oplus (\eta_q' \otimes \lambda), \mu_q),$$

where λ is the usual line bundle over the projective space bundle $RP(\nu \oplus R)$, where μ is given by $\mu(T_1, [v, s]_p) = [-v, s]_p$ and

$$\mu(T_j, [v, s]_p) = [\overline{\psi}(T_{j-1}, v), s]_{\psi(p)}$$

if $2 \leq j \leq k$, and where, for $1 \leq i \leq q$, μ_i is given by

$$\mu_i(T_1, ((u, w \otimes r), [v, s]_p)) = ((u, -w \otimes r), [-v, s]_p)$$

and

$$\mu_i(T_j, ((u, w \otimes r), [v, s]_p)) = ((\psi_i(T_{j-1}, u), (\psi_i'(T_{j-1}, w)) \otimes r), [\overline{\psi}(T_{j-1}, v), s])_{\psi(p)}$$

if $2 \leq j \leq k$. Here $[v, s]_p$ denotes a typical element of the total space of $RP(\nu \oplus R)$ belonging to the fiber over the point $p \in N$, and $((u, w \otimes r), [v, s]_p)$ denotes an element of the total space of $\eta_i \oplus (\eta_i' \otimes \lambda)$ belonging to the fiber over $[v, s]_p$ (we are suppressing all bundle maps). Denote by α' the class of this $((Z_2)^k, q)$-manifold-bundle in $\mathcal{N}_{m,n_1,\ldots,n_q}((Z_2)^k, q)$. Then $F(\alpha') = \beta + \gamma$, where

$$\gamma = [(RP(\nu), \mu, \lambda, \mu^*, (\kappa_i, \mu_i), (\kappa_i', \mu_i'))],$$

with μ and λ being restrictions of the previous μ, λ, where μ^* is induced by μ, $\kappa_i = \eta_i \oplus (\eta_i' \otimes R)$, κ_i' is the trivial zero bundle and μ_i is induced by the previous μ_i. Since $0 = SF(\alpha') = S(\beta) + S(\gamma) = S(\gamma)$ and S maps the summand

$$\mathcal{N}_{m-1,1,n_1,0,\ldots,n_q,0}((Z_2)^{k-1}, 2q+1)$$

isomorphically onto

$$\widehat{\mathcal{N}}_{m-1,n_1,\ldots,n_q}((Z_2)^k, q),$$

one obtains $\gamma = 0$ and so $\alpha = \alpha'$. □

For the next lemma we adopt a slightly different notation for the monomorphism F of the fixed point exact sequence; put

$$F : \mathcal{N}_{m,n_1,\ldots,n_q}((Z_2)^k, q) \to \bigoplus \mathcal{N}_{r,l_1,\ldots,l_{2q+1}}((Z_2)^{k-1}, 2q+1),$$

where the sum is then over all sequences with $r + l_1 = m$ and $l_{2i} + l_{2i+1} = n_i$, $1 \leq i \leq q$.

Lemma 3. *Let*

$$\beta = \sum [(N, \Psi, \eta_1, \Psi_1, ..., \eta_{2q+1}, \Psi_{2q+1})]$$

be an element of $\bigoplus \mathcal{N}_{r,l_1,\ldots,l_{2q+1}}((Z_2)^{k-1}, 2q+1)$ *which belongs to the image of* F, *with* $\eta_{2p} = \eta_{2p}' \oplus R$ *for some* $1 \leq p \leq q$ *and with the action* Ψ_{2p} *of* $(Z_2)^{k-1}$ *on* $E(\eta_{2p})$ *being of the form*

$$\Psi_{2p}(T_j, (v_{2p}', r)) = (\Psi_{2p}'(T_j, v_{2p}'), r)$$

for all $v_{2p}' \in E(\eta_{2p}')$ *(this condition over* η_{2p} *is required for each summand).*

a) If $F(\alpha) = \beta$, *we can choose a representative* $\bigcup (M, \mu, \xi_1, \mu_1, ..., \xi_q, \mu_q)$ *for* α *such that* ξ_p *is of the form* $\xi_p' \oplus R$ *and the action* μ_p *of* $(Z_2)^k$ *on* $E(\xi_p)$ *is trivial on the trivial 1-dimensional factor.*

b) $\beta' = \sum[(N, \Psi, \eta_1, \Psi_1, ..., \eta_{2p-1}, \Psi_{2p-1}, \eta'_{2p}, \Psi'_{2p}, \eta_{2p+1}, \Psi_{2p+1}, ..., \eta_{2q+1}, \Psi_{2q+1})]$ *belongs to the image of*

$$F\colon \mathcal{N}_{m,n_1,...,n_{p-1},n_p-1,n_{p+1},...,n_q}((Z_2)^k, q) \to \bigoplus \mathcal{N}_{r,l_1,...,l_{2p-1},l_{2p}-1,l_{2p+1},...,l_{2q+1}}((Z_2)^{k-1}, 2q+1).$$

Proof. a) Let $\bigcup(M, \mu, \xi_1, \mu_1, ..., \xi_q, \mu_q)$ be the representative of α given by Lemma 2. Note then that $\xi_p = \eta_{2p} \oplus (\eta_{2p+1} \otimes \lambda) = (\eta'_{2p} \oplus (\eta_{2p+1} \otimes \lambda)) \oplus R = \xi'_p \oplus R$, where λ denotes the line bundle over $RP(\eta_1 \oplus R)$. The hypothesis on Ψ_{2p} implies additionally that μ_p acts trivially on the trivial 1-dimensional factor of ξ_p.

b) One has a commutative diagram

(∗)

$$\begin{array}{ccc} \mathcal{N}_{m,n_1,...,n_q}((Z_2)^k, q) & \xrightarrow{F} & \bigoplus \mathcal{N}_{r,l_1,...,l_{2q+1}}((Z_2)^{k-1}, 2q+1) \\ \uparrow I_* & & \uparrow I_* \\ \mathcal{N}_{m,n_1,...,n_{p-1},n_p-1,n_{p+1},...,n_q}((Z_2)^k, q) & \xrightarrow{F} & \bigoplus \mathcal{N}_{r,l_1,...,l_{2p-1},l_{2p}-1,l_{2p+1},...,l_{2q+1}}((Z_2)^{k-1}, 2q+1) \end{array}$$

where the left I_* adds a trivial line bundle to the p-th factor and extends the previous $(Z_2)^k$-action on this factor so that it acts trivially on that trivial line bundle, and where the right I_* adds a trivial line bundle to the $2p$-th factor and extends the previous $(Z_2)^{k-1}$-action in the same way. It is then clear that $I_*(\beta') = \beta$. Putting $\alpha' = \sum[(M, \mu, \xi_1, \mu_1, ..., \xi_{p-1}, \mu_{p-1}, \xi'_p, \mu_{p|\xi'_p}, \xi_{p+1}, \mu_{p+1}, ..., \xi_q, \mu_q)]$, one has by a) that $I_*(\alpha') = \alpha$. Since I_* is easily seen to be a monomorphism (by looking at fixed sets), one concludes that $F(\alpha') = \beta'$. □

Remark. The importance of the occurrence of a bundle with a trivial 1-dimensional factor in an even position is clear from the above proof.

We now proceed to prove our theorem. We start with a Whitney sum $\bigoplus \varepsilon_a \to F$ which is the fixed data of a $(Z_2)^k$-action (M^n, Φ), $\Phi = (T_1, T_2, ..., T_k)$, with $\varepsilon_{a_0} = \varepsilon'_{a_0} \oplus R$ for some fixed a_0, and we wish to show that the Whitney sum obtained from $\bigoplus \varepsilon_a$ by replacing ε_{a_0} by ε'_{a_0} is also the fixed data of some $(Z_2)^k$-action. Here F is not assumed to be connected, and this fact must be taken into account; thus, we are supposing $\varepsilon_{a_0|F_i} = \varepsilon'_{a_0|F_i} \oplus R$ (which implies $\dim(F_i) < n$) and

$$\dim(F_i) + (\sum_a \dim(\varepsilon_{a|F_i})) = n$$

for each component F_i of F.

For $1 \le t \le k$, consider an arbitrary partition $m + n_1 + ... + n_{2^{t-1}-1} = n$ of n with 2^{t-1} terms, and set $\tau_{t-1} = (m, n_1, ..., n_{2^{t-1}-1})$. When $t = 1$ we are considering the trivial partition $\tau_0 = (n)$. For each τ_{t-1} one has the fixed point monomorphism

$$F_{\tau_{t-1}} : \mathcal{N}_{m,n_1,...,n_{2^{t-1}-1}}((Z_2)^{k-t+1}, 2^{t-1} - 1) \to \bigoplus \mathcal{N}_{r,l_1,...,l_{2^t-1}}((Z_2)^{k-t}, 2^t - 1)$$

with the sum being over all sequences with $r + l_1 = m$, $l_{2i} + l_{2i+1} = n_i$, $1 \le i \le 2^{t-1} - 1$. Taking the sum over all such partitions, one obtains the monomorphism

$$F_t = \bigoplus_{\tau_{t-1}} F_{\tau_{t-1}} : \bigoplus_{\tau_{t-1}} \mathcal{N}_{m,n_1,...,n_{2^{t-1}-1}}((Z_2)^{k-t+1}, 2^{t-1} - 1) \to \bigoplus_{\tau_{t-1}} \bigoplus \mathcal{N}_{r,l_1,...,l_{2^t-1}}((Z_2)^{k-t}, 2^t - 1).$$

Note that

$$\bigoplus_{\tau_{t-1}}\bigoplus \mathcal{N}_{r,l_1,...,l_{2^t-1}}((Z_2)^{k-t},2^t-1)=\bigoplus_{\tau_t}\mathcal{N}_{r,l_1,...,l_{2^t-1}}((Z_2)^{k-t},2^t-1),$$

where as above τ_t runs over all partitions $r+l_1+...+l_{2^t-1}=n$ of n with 2^t terms. That is, we can rewrite

$$F_t:\bigoplus_{\tau_{t-1}}\mathcal{N}_{m,n_1,...,n_{2^{t-1}-1}}((Z_2)^{k-t+1},2^{t-1}-1)\to\bigoplus_{\tau_t}\mathcal{N}_{r,l_1,...,l_{2^t-1}}((Z_2)^{k-t},2^t-1).$$

Note that we are considering $F_1:\mathcal{N}_n((Z_2)^k)\to\bigoplus_{r+s=n}\mathcal{N}_{r,s}((Z_2)^{k-1},1)$ as the initial fixed point monomorphism.

Following the same principle, for $1\le t\le k$ one has monomorphisms

$$F'_t:\bigoplus_{\tau'_{t-1}}\mathcal{N}_{m',n'_1,...,n'_{2^{t-1}-1}}((Z_2)^{k-t+1},2^{t-1}-1)\to\bigoplus_{\tau'_t}\mathcal{N}_{r',l'_1,...,l'_{2^t-1}}((Z_2)^{k-t},2^t-1),$$

where τ'_{t-1} and τ'_t run over all partitions of $n-1$ with 2^{t-1} and 2^t terms, respectively.

Now $\bigoplus\varepsilon_a\to F$ represents a bordism class β of $\bigoplus_{\tau_k}\mathcal{N}_{r,l_1,...,l_{2^k-1}}((Z_2)^0,2^k-1)$. By using Lemma 1 we can suppose first, with no loss, that ε_{a_0} occupies the 2^{k-1}-th position; in other words, considering the order prefixed in Section 2 for the nontrivial representations of $(Z_2)^k$, we are supposing that $a_0=(1,0,0,...,0)$. Denote by β' the bordism class of the Whitney sum obtained from $\bigoplus\varepsilon_a$ by replacing ε_{a_0} by ε'_{a_0}. Rewriting $\bigoplus\varepsilon_a$ as $\bigoplus_{h=1}^{2^k-1}\varepsilon_h$, we note that β can be written as $\beta=\sum_{\tau_{k-1}}\beta_{\tau_{k-1}}$, where the sum is over all partitions τ_{k-1} of n with 2^{k-1} terms and, for each $\tau_{k-1}=(m,n_1,...,n_{2^{k-1}-1}),\beta_{\tau_{k-1}}$ is represented by the union of the components $\bigoplus\varepsilon_h\to F_i$ which satisfy $\dim(F_i)+\dim(\varepsilon_1)=m$, $\dim(\varepsilon_{2l})+\dim(\varepsilon_{2l+1})=n_l$, $1\le l\le 2^{k-1}-1$. In a similar way we can write $\beta'=\sum_{\tau'_{k-1}}\beta'_{\tau'_{k-1}}$.

Now by hypothesis $F_kF_{k-1}...F_1[(M^n,\Phi)]=\beta$. By dimensional considerations involving the definition of F and the fact that $F_k=\bigoplus_{\tau_{k-1}}F_{\tau_{k-1}}$, one sees then that $\alpha=F_{k-1}...\,...F_1[(M^n,\Phi)]$ can be written as $\alpha=\sum_{\tau_{k-1}}\alpha_{\tau_{k-1}}$, where, for each $\tau_{k-1}=(m,n_1,...,n_{2^{k-1}-1})$, $\alpha_{\tau_{k-1}}$ belongs to $\mathcal{N}_{m,n_1,...,n_{2^{k-1}-1}}(Z_2,2^{k-1}-1)$ and $F_{\tau_{k-1}}(\alpha_{\tau_{k-1}})=\beta_{\tau_{k-1}}$. Using Lemma 3 for $F_{\tau_{k-1}}$ one has that each $\alpha_{\tau_{k-1}}$ can be represented by a $(Z_2,2^{k-1}-1)$-manifold-bundle so that the bundle occurring in the 2^{k-2}-th position contains a trivial 1-dimensional factor on which Z_2 acts trivially, and so the same is valid for α. Additionally one obtains that $F'_k(\alpha')=\beta'$, where α' is obtained from α by omitting the trivial 1-dimensional factor of the 2^{k-2}-th position.

Now we use induction on t, $2\le t\le k$, replacing β by $F_tF_{t-1}...F_1[(M^n,\Phi)]$. We emphasize that the initial choice of the 2^{k-1}-th position and part a) of Lemma 3 allow the use of this argument in each step of the induction. In this way $F_1[(M^n,\Phi)]$ is an element of $\bigoplus_{r+s=n}\mathcal{N}_{r,s}((Z_2)^{k-1},1)$ of the form $\gamma=\sum[(N,\Psi,\eta\oplus R,\Psi^*)]$ with Ψ^* acting trivially on the trivial 1-dimensional factor, and $F'_kF'_{k-1}...F'_2(\gamma')=\beta'$, where $\gamma'=\sum[(N,\Psi,\eta,\Psi^*_{|\eta})]$.

To proceed with the proof, we need first to recall a fact proved in [6, Section 3]. Let $(V, T_1, T_2, ..., T_k)$ be a $(Z_2)^k$-action with fixed data $\bigoplus_a \varepsilon_a \to F$, and let $\mu \to F_{T_1}$ be the normal bundle of F_{T_1} in V. Consider the $(Z_2)^{k-1}$-action $(RP(\mu \oplus R), T_2', ..., T_k')$, where each T_i' is induced by the involution on $\mu \oplus R$ given by the action induced by T_i in the fibers of μ and the trivial action in R. Then the fixed data of this action is explicitly constructible from $\bigoplus_a \varepsilon_a \to F$; specifically, denoting by $b = (b_1, b_2, ..., b_{k-1})$ and $c = (c_1, c_2, ..., c_{k-1})$ arbitrary representations of $(Z_2)^{k-1}$, the fixed set of the above action is

$$\mathcal{F} = RP(\varepsilon_{(1,0,0,...,0)} \oplus R) \cup (\bigcup_{b \neq (0,...,0)} RP(\varepsilon_{(1,b)})),$$

where each such component fibers over F, and the fixed data in question is $\bigoplus_{c \neq (0,...,0)} \varepsilon_c \to \mathcal{F}$, where over $RP(\varepsilon_{(1,0,0,...,0)} \oplus R)$ one has $\varepsilon_c = (\lambda \otimes \varepsilon_{(1,c)}) \oplus \varepsilon_{(0,c)}$, and over $RP(\varepsilon_{(1,b)})$ one has $\varepsilon_c = (\lambda \otimes (\varepsilon_{(1,0,0,...,0)} \oplus R)) \oplus \varepsilon_{(0,b)}$ if $c = b$ and $\varepsilon_c = ((\lambda \otimes \varepsilon_{(1,b+c)})) \oplus \varepsilon_{(0,c)}$ if $c \neq b$ (here the $\varepsilon_{(x,c)}$ and $\varepsilon_{(x,b)}$ are pulled back from F, λ means the line bundle over the specific projective space bundle, and $b + c = (b_1 + c_1, ..., b_{k-1} + c_{k-1})$, where $b_j + c_j$ is taken modulo 2).

Taking into account that $F_1[(V, T_1, T_2, ..., T_k)] = [(F_{T_1}, T_2, ..., T_k; \mu, T_2^*, ..., T_k^*)]$, T_i^* being the action induced by T_i in the fibers of μ, the proof of the above fact tells us particularly that if $(N, T_2, ..., T_k; \eta, T_2^*, ..., T_k^*)$ represents a class belonging to $\bigoplus_{r+s=n} \mathcal{N}_{r,s}((Z_2)^{k-1}, 1)$ and has fixed data $\bigoplus_a \varepsilon_a \to F$, then the fixed data of $(RP(\eta \oplus R), T_2', ..., T_k')$, where T_i' is induced by the involution given by T_i^* on η and the trivial action on R, is exactly as described above.

Using the same argument used to prove the above fact, we get the following:

Proposition. *Let $(N, T_2, ..., T_k; \eta, T_2^*, ..., T_k^*)$ be a representative of a class belonging to $\bigoplus_{r+s=n-1} \mathcal{N}_{r,s}((Z_2)^{k-1}, 1)$ such that its fixed data is $\bigoplus_a \varepsilon_a \to F$. Consider the corresponding $(Z_2)^{k-1}$-action $(RP(\eta), T_2', ..., T_k')$, T_i' being induced from T_i^*. Then the fixed set of this action is $\mathcal{F}' = RP(\varepsilon_{(1,0,0,...,0)}) \cup (\bigcup_{b \neq (0,...,0)} RP(\varepsilon_{(1,b)}))$ fibering over F, and the fixed data is $\bigoplus_{c \neq (0,...,0)} \varepsilon_c' \to \mathcal{F}'$, where over $RP(\varepsilon_{(1,0,0,...,0)})$ one has $\varepsilon_c' = (\lambda \otimes \varepsilon_{(1,c)}) \oplus \varepsilon_{(0,c)}$, and over $RP(\varepsilon_{(1,b)})$ one has $\varepsilon_c' = (\lambda \otimes \varepsilon_{(1,0,0,...,0)}) \oplus \varepsilon_{(0,b)}$ if $c = b$ and $\varepsilon_c' = (\lambda \otimes \varepsilon_{(1,b+c)}) \oplus \varepsilon_{(0,c)}$ if $c \neq b$.*

We return to our proof. Consider the fixed point sequences

$$0 \to \mathcal{N}_n((Z_2)^k, 0) \xrightarrow{F_1} \bigoplus_{r+s=n} \mathcal{N}_{r,s}((Z_2)^{k-1}, 1) \xrightarrow{S} \widehat{\mathcal{N}}_{n-1}((Z_2)^k, 0) \to 0$$

and

$$0 \to \mathcal{N}_{n-1}((Z_2)^k, 0) \xrightarrow{F_1'} \bigoplus_{r+s=n-1} \mathcal{N}_{r,s}((Z_2)^{k-1}, 1) \xrightarrow{S'} \widehat{\mathcal{N}}_{n-2}((Z_2)^k, 0) \to 0$$

and denote by ρ (respectively ρ') the inverse for S (respectively S') on the summand $\mathcal{N}_{n-1,1}((Z_2)^{k-1}, 1)$ (respectively $\mathcal{N}_{n-2,1}((Z_2)^{k-1}, 1)$). By the description of S and

ρ one has

$$\rho S(\gamma) = \rho S(\sum[(N, \Psi, \eta \oplus R, \Psi^*)]) = \rho S F_1[(M^n, \Phi)]$$
$$= [(RP(\eta \oplus R), T_2', ..., T_k'; \lambda, T_2^*, ..., T_k^*)],$$

where T_i' is defined from T_i as above and T_i^* is induced by T_i'. Now we have proved before that $F_k' F_{k-1}' ... F_2'(\gamma') = \beta'$, that is, that $\bigcup(N, \Psi, \eta, \Psi^*_{|\eta})$ has as fixed data the Whitney sum obtained from $\bigoplus_a \varepsilon_a$ by replacing ε_{a_0} by ε'_{a_0}. Therefore the fixed data of $(RP(\eta \oplus R), T_2', ..., T_k')$, $\bigoplus_{c \neq (0,...,0)} \varepsilon_c$, is exactly as described by the above fact with ε'_{a_0} in the place of ε_{a_0}, recalling that $a_0 = (1, 0, .., 0)$. In the same way, the fixed data of $(RP(\eta), T_2', ..., T_k')$, with each T_i' being restriction of the previous T_i', is $\bigoplus_{c \neq (0,..,0)} \varepsilon'_c \to \mathcal{F}'$, where ε'_c and $\mathcal{F}'$ are given by the above proposition with ε'_{a_0} in the place of ε_{a_0}.

On the other hand, we can see that under the map $F_k F_{k-1} ... F_2$ the line bundle $\lambda \to RP(\eta \oplus R)$ is decomposed into a Whitney sum $\bigoplus_{i=1}^{2^{k-1}} z_i$ consisting of $2^{k-1} - 1$ zero bundles and with one factor being the specific line bundle over each component of the fixed set $\mathcal{F}$ of $(RP(\eta \oplus R), T_2', ..., T_k')$. We conclude then in principle that the fixed data of $(RP(\eta \oplus R), T_2', ..., T_k'; \lambda, T_2^*, ..., T_k^*)$ has the form $(\bigoplus_{c \neq (0,...,0)} \varepsilon_c) \oplus (\bigoplus_{i=1}^{2^{k-1}} z_i) \to \mathcal{F}$, but we need to describe this fixed data more precisely, and this must be done in terms of the nontrivial representations of $(Z_2)^k$. To do that, write the fixed data in question as

$$\bigoplus_{a \neq (0,..,0)} \xi_a \to \mathcal{F} = (\bigoplus_{c \neq (0,..,0)} \xi_{(1,c)}) \oplus \xi_{a_0} \oplus (\bigoplus_{c \neq (0,..,0)} \xi_{(0,c)}) \to \mathcal{F},$$

and take first $b \neq (0, .., 0)$. We see above that, over $RP(\varepsilon_{(1,b)})$,

$$\varepsilon_c = (\lambda \otimes (\varepsilon'_{a_0} \oplus R)) \oplus \varepsilon_{(0,b)}$$

if $c = b$ and

$$\varepsilon_c = (\lambda \otimes \varepsilon_{(1,b+c)}) \oplus \varepsilon_{(0,c)}$$

if $c \neq b$. Observe that T_1 acts as -1 on λ, -1 on $\varepsilon_{a_0} = \varepsilon'_{a_0} \oplus R$ and 1 on $\varepsilon_{(0,b)}$, hence T_1 acts as 1 on ε_b. Since additionally $(Z_2)^{k-1}$ (generated by $T_2, ..., T_k$) acts as b on ε_b, this means that

$$\xi_{(0,b)} = \varepsilon_b = (\lambda \otimes (\varepsilon'_{a_0} \oplus R)) \oplus \varepsilon_{(0,b)} = ((\lambda \otimes \varepsilon'_{a_0}) \oplus \varepsilon_{(0,b)}) \oplus \lambda$$

over $RP(\varepsilon_{(1,b)})$. For $c \neq b$ one has that T_1 acts as -1 on $\varepsilon_{(1,b+c)}$, 1 on $\varepsilon_{(0,c)}$, and again as -1 on λ, hence T_1 acts as 1 on ε_c; since $(Z_2)^{k-1}$ acts as c on ε_c, one has then that $\xi_{(0,c)} = \varepsilon_c = (\lambda \otimes \varepsilon_{(1,b+c)}) \oplus \varepsilon_{(0,c)}$ over $RP(\varepsilon_{(1,b)})$.

These facts imply that there exists a one-one correspondence between the collection formed by the bundles $\xi_{a_0} \to RP(\varepsilon_{(1,b)})$ and $\xi_{(1,c)} \to RP(\varepsilon_{(1,b)})$, $c \neq (0, .., 0)$, and the collection of the bundles $z_i \to RP(\varepsilon_{(1,b)})$, $1 \leq i \leq 2^{k-1}$. Since T_1 acts as -1 on λ and $(Z_2)^{k-1}$ acts as b on λ, one has then that $\xi_{(1,b)} = \lambda \to RP(\varepsilon_{(1,b)})$, while ξ_{a_0} and $\xi_{(1,c)}$ for $c \neq (0, .., 0)$ and $c \neq b$ are zero bundles over $RP(\varepsilon_{(1,b)})$.

Next we must analyse the fixed data over $RP(\varepsilon'_{a_0} \oplus R)$. In this case one has $\varepsilon_c = (\lambda \otimes \varepsilon_{(1,c)}) \oplus \varepsilon_{(0,c)}$. Since T_1 acts as -1 on λ, -1 on $\varepsilon_{(1,c)}$, and 1 on $\varepsilon_{(0,c)}$, and

$(Z_2)^{k-1}$ acts as c on ε_c, one has that $(Z_2)^k$ acts as $(0,c)$ on ε_c for each $c \neq (0,..,0)$; therefore $\xi_{(0,c)} = \varepsilon_c = (\lambda \otimes \varepsilon_{(1,c)}) \oplus \varepsilon_{(0,c)}$ over $RP(\varepsilon'_{a_0} \oplus R)$. Similarly as above, one also has that $(Z_2)^{k-1}$ acts trivially on λ and again T_1 acts as -1 on λ; hence $\xi_{a_0} = \lambda \to RP(\varepsilon'_{a_0} \oplus R)$ while each $\xi_{(1,c)}$, $c \neq (0,..,0)$, is the zero bundle over $RP(\varepsilon'_{a_0} \oplus R)$. This completes the desired description.

In the same way one has that the fixed data of $(RP(\eta), T'_2, ..., T'_k; \lambda, T^*_2, ..., T^*_k)$, where T'_i, T^*_i and λ are restrictions of the previous T'_i, T^*_i and λ, has the form $(\bigoplus_{c \neq (0,..,0)} \varepsilon'_c) \oplus (\bigoplus_{i=1}^{2^{k-1}} z'_i) \to \mathcal{F}'$, where $\bigoplus_{c \neq (0,..,0)} \varepsilon'_c \to \mathcal{F}'$ is given by the above proposition with ε'_{a_0} in the place of ε_{a_0} and where $\bigoplus_{i=1}^{2^{k-1}} z'_i$ consists of $2^{k-1} - 1$ zero bundles and one factor equal to the line bundle over each component of $\mathcal{F}'$. Writing this fixed data as

$$\bigoplus_{a \neq (0,..,0)} \xi'_a \to \mathcal{F}' = (\bigoplus_{c \neq (0,..,0)} \xi'_{(1,c)}) \oplus \xi'_{a_0} \oplus (\bigoplus_{c \neq (0,..,0)} \xi'_{(0,c)}) \to \mathcal{F}',$$

one has similarly as above that, for $b \neq (0,..,0)$, $\xi'_{(0,b)} = \varepsilon'_b = (\lambda \otimes \varepsilon'_{a_0}) \oplus \varepsilon_{(0,b)}$, $\xi'_{(0,c)} = \varepsilon'_c = (\lambda \otimes \varepsilon_{(1,b+c)}) \oplus \varepsilon_{(0,c)}$ for $c \neq b$, $\xi'_{(1,b)} = \lambda$, $\xi'_{a_0} = 0$ and $\xi'_{(1,c)} = 0$ for $c \neq (0,..,0)$ and $c \neq b$ (here the bundles are considered over $RP(\varepsilon_{(1,b)})$); and over $RP(\varepsilon'_{a_0})$ one has $\xi'_{(0,c)} = \varepsilon'_c = (\lambda \otimes \varepsilon_{(1,c)}) \oplus \varepsilon_{(0,c)}$ for each $c \neq (0,..,0)$, $\xi'_{a_0} = \lambda$ and $\xi'_{(1,c)} = 0$ for all $c \neq (0,..,0)$, which describes the fixed data in question.

Since

$$[\bigoplus_a \xi_a \to \mathcal{F}] = F_k F_{k-1} ... F_2 [(RP(\eta \oplus R), T'_2, ..., T'_k; \lambda, T^*_2, ..., T^*_k)]$$
$$= F_k F_{k-1} ... F_2 \rho S F_1 [(M^n, \Phi)] = 0,$$

one has that all characteristic numbers of $\bigoplus_a \xi_a \to \mathcal{F}$ are zero. This fact implies that all characteristic numbers of $\bigoplus_a \xi'_a \to \mathcal{F}'$ must also be zero, as we will see next.

Before proceeding we need to establish a general fact. Consider formal classes $w_0 = 1, w_1, w_2, ..., w_l, W_0 = 1, W_1, W_2, ..., W_l, c$, where w_i and W_i have degree i and c has degree 1, subject to the modulo 2 relations $W_i = w_i + c w_{i-1}$, $1 \le i \le l$ (putting $w = 1 + w_1 + w_2 + ... + w_l$, $W = 1 + W_1 + W_2 + ... + W_l + W_{l+1}$, this comes from $W = (1+c)w$, considering that $W_{l+1} = c w_l$ plays no role in our considerations). One has $w_1 = W_1 + c$, $w_2 = W_2 + c w_1 = W_2 + c W_1 + c^2$, and inductively $w_j = \sum_{s=0}^{j} W_s c^{j-s}$ for $1 \le j \le l$. For a sequence of natural numbers $\omega = (i_1, i_2, .., i_t)$ one has, using these expressions, that $w_{i_1} w_{i_2} ... w_{i_t}$ is a homogeneous polynomial over Z_2 of degree $|\omega| = i_1 + i_2 + ... + i_t$ involving only c and, $W_1, W_2, ..., W_l$ and which depends only on ω; we denote it by

$$P_\omega(c, W_1, W_2, ..., W_l) = P_\omega(c, W).$$

Now consider a collection of closed manifolds $F^0, F^1, ..., F^l$, where each F^i is a disjoint union of closed manifolds of dimension i, and suppose that η_i, μ_i are respectively real vector bundles over F_i and $RP(\eta_i)$ such that $\dim(\eta_i) = l + 1 - i$ and $\dim(\mu_i)$ is constant for $0 \le i \le l$. Denote by λ_i the line bundle over $RP(\eta_i)$, by

c_i the class of λ_i, by $j_i : RP(\eta_i) \to RP(\eta_i \oplus R)$ the inclusion, and write $\eta = \bigcup_i \eta_i$, $\lambda \to RP(\eta) = \bigcup_i (\lambda_i \to RP(\eta_i))$, $\mu \to RP(\eta) = \bigcup_i (\mu_i \to RP(\eta_i))$. Then we have

Fact. *a) Take $\omega = (i_1, i_2, ..., i_t)$ and j with $|\omega| + j = l$. Then the Whitney number $c^j W_\omega(RP(\eta))\ [RP(\eta)]$ of $\lambda \to RP(\eta)$ is equal to the Whitney number $c^{j+1} P_\omega(c, W(RP(\eta \oplus R)))[RP(\eta \oplus R)]$ of $\lambda \to RP(\eta \oplus R)$.*

b) Consider $\omega = (i_1, i_2, ..., i_t)$, $\rho = (s_1, s_2, ..., s_q)$ and j with $|\omega| + |\rho| + j = l$. Then the Whitney number $c^j W_\omega(RP(\eta)) V_\rho(\mu)[RP(\eta)]$ of $\lambda \oplus \mu \to RP(\eta)$ is equal to the Whitney number $c^j W_\omega(RP(\eta)) P_\rho(c, V(\mu \oplus \lambda))[RP(\eta)]$ of $\lambda \oplus (\mu \oplus \lambda) \to RP(\eta)$ (here we are considering Whitney numbers associated to Whitney sums of two bundles).

Proof. a) One has for each i that $j_i^*(W(RP(\eta_i \oplus R))) = (1 + c_i) W(RP(\eta_i))$; hence $j_i^*(W_r(RP(\eta_i \oplus R))) = W_r(RP(\eta_i)) + c_i W_{r-1}(RP(\eta_i))$ for $1 \le r \le l$. In this way

$$\begin{aligned} c_i^j W_\omega(RP(\eta_i))[RP(\eta_i)] &= c_i^j j_i^*(P_\omega(c_i, W(RP(\eta_i \oplus R))))[RP(\eta_i)] \\ &= c_i^{j+1} P_\omega(c_i, W(RP(\eta_i \oplus R)))[RP(\eta_i \oplus R)], \end{aligned}$$

and this relation does not depend on $0 \le i \le l$; that is, it is formally the same for the several i's. Since Whitney numbers are additive, the fact follows.

b) Since $W(\mu_i \oplus \lambda_i) = (1 + c_i) W(\mu_i)$, the proof follows the same lines of a). □

Returning again to our proof, observe first that $[\bigoplus_a \xi_a]$ belongs to

$$\bigoplus \mathcal{N}_l(BO(j_1) \times ... \times BO(j_{2^{k-1}}) \times BO(r_1) \times ... \times BO(r_{2^{k-1}-1})),$$

where $j_i = 0$ for all i with the exception of one i_0 for which $j_{i_0} = 1$, and where the sum is taken over all such sequences with $l + 1 + \sum r_i = n$. A similar description is valid for $[\bigoplus_a \xi'_a]$ with $n - 1$ in the place of n. Here the j_i's correspond to the 2^{k-1} representations of $(Z_2)^k$ of type $(1, c)$ (particularly, $j_{2^{k-1}}$ corresponds to $a_0 = (1, 0, .., 0)$), while the r_i's correspond to the representations $(0, c)$, $c \ne (0, .., 0)$. There is a one-one correspondence between components of $\bigoplus \xi_a$ and $\bigoplus \xi'_a$ modulo components of $\bigoplus \xi_a$ coming from components F^i of F over which $\varepsilon'_{a_0} = 0$ (because for these components $RP(\varepsilon'_{a_0} \oplus R) = F^i$ contributes to $\bigoplus \xi_a$, while $RP(\varepsilon'_{a_0}) = \emptyset$ does not contribute to $\bigoplus \xi'_a$). In fact, for a component of $\bigoplus \xi'_a$ of type $\bigoplus \xi'_a \to RP(\varepsilon'_{a_0})$ with $\varepsilon'_{a_0} \ne 0$ and coming from F^i, one has the corresponding component of $\bigoplus \xi_a$ given by $\bigoplus \xi_a \to RP(\varepsilon'_{a_0} \oplus R)$ coming from the same component F^i and with each ξ'_a being the restriction on $RP(\varepsilon'_{a_0})$ of the corresponding ξ_a; and for a component of $\bigoplus \xi'_a$ of type $\bigoplus \xi'_a \to RP(\varepsilon_{(1,b)})$ for some $b \ne (0, .., 0)$ and coming from F^i, one has the corresponding component of $\bigoplus \xi_a$ given by $\bigoplus \xi_a \to RP(\varepsilon_{(1,b)})$ coming from the same F^i, with $\xi_{(0,b)} = \xi'_{(0,b)} \oplus \lambda$ and $\xi_a = \xi'_a$ for all $a \ne (0, b)$. It is important to note also that if $b \ne c$ then terms of type $[\bigoplus \xi_a \to RP(\varepsilon_{(1,b)})]$ and $[\bigoplus \xi_a \to RP(\varepsilon_{(1,c)})]$ ($[\bigoplus \xi_a \to RP(\varepsilon'_{a_0} \oplus R)]$ when $c = (0, .., 0)$) belong to different summands since the corresponding line bundles λ occur in different positions (the same can be said about terms of type $[\bigoplus \xi'_a \to RP(\varepsilon_{(1,b)})]$ and $[\bigoplus \xi'_a \to RP(\varepsilon_{(1,c)})]$); this is the crucial point for comparing characteristic numbers of $\bigoplus \xi_a$ and $\bigoplus \xi'_a$.

The fact $[\bigoplus \xi_a] = 0$ means that the part of $[\bigoplus \xi_a]$ belonging to a given summand $\mathcal{N}_l(BO(j_1) \times ... \times BO(j_{2^{k-1}}) \times BO(r_1) \times ... \times BO(r_{2^{k-1}-1}))$ is zero for each such summand, and one wants to show the same for $[\bigoplus \xi'_a]$. Then fix a summand with

$l+1+\sum r_i = n-1$ and for which $j_{2^{k-1}}$ (corresponding to $a_0 = (1,0,..,0)$) is 1. The part of $[\bigoplus \xi'_a]$ that belongs to this summand comes from the collection $L = \bigcup_{i=0}^{l} F^i$ of components of F, where each F^i is the union of all i-dimensional components of F over which $\dim(\varepsilon'_{a_0}) = l+1-i$ and $\dim(\xi'_{(0,c)})$ over $RP(\varepsilon'_{a_0})$ is r_j when c occupies the j-th position; this part is $[\bigoplus \xi'_a \to RP(\varepsilon'_{a_0})_{|L}]$, observing that λ occupies in this case the 2^{k-1}-th position. By the above comments one then has that the part of $[\bigoplus \xi_a]$ that belongs to the summand $\mathcal{N}_{l+1}(BO(j_1) \times ... \times BO(j_{2^{k-1}}) \times BO(r_1) \times ... \times BO(r_{2^{k-1}-1}))$ consists of $[\bigoplus \xi_a \to RP(\varepsilon'_{a_0} \oplus R)_{|L}]$ (with each ξ'_a being restriction of the corresponding ξ_a) plus a term coming from the union G of all $l+1$-dimensional components F^{l+1} of F over which $\dim(\varepsilon'_{a_0}) = 0$ and $\dim(\xi_{(0,c)})$ over $RP(\varepsilon'_{a_0} \oplus R) = F^{l+1}$ is r_j when c occupies the j-th position; that is, this part is

$$[\bigoplus \xi_a \to RP(\varepsilon'_{a_0} \oplus R)_{|L}] + [\bigoplus \xi_a \to G]$$

A general Whitney number of $\bigoplus \xi'_a \to RP(\varepsilon'_{a_0})_{|L}$ is of the form

$$c^t W_\omega(RP(\varepsilon'_{a_0})_{|L}) K[RP(\varepsilon'_{a_0})_{|L}],$$

where K is a product of classes of the several $\xi'_{(0,c)}$ and $t + |\omega| + \deg(K) = l$ (evidently this means a sum of such Whitney numbers corresponding to the several components). By Fact a) this number is equal to

$$c^{t+1} P_\omega(c, W(RP(\varepsilon'_{a_0} \oplus R)_{|L})) K[RP(\varepsilon'_{a_0} \oplus R)_{|L}],$$

which in his turn is equal to $c^{t+1} P_\omega(c, W(G)) K[G]$ since the part of $[\bigoplus \xi_a]$ in question is zero. But the line bundle over G is the trivial one-dimensional bundle $R \to G$; hence its class c is zero. Since $t+1 \geq 1$, the above number is zero; therefore the part of $[\bigoplus \xi'_a]$ in question is zero.

Now take a summand with $l+1+\sum r_i = n-1$ for which $j_s = 1$ for $s \neq 2^{k-1}$, considering then that j_s corresponds to some representation $(1,b)$ with $b \neq (0,...,0)$. The part of $[\bigoplus \xi'_a]$ that belongs to this summand comes from the collection $L = \bigcup_{i=0}^{l} F^i$ of components of F, where each F^i is the union of all i-dimensional components of F over which $\dim(\varepsilon_{(1,b)}) = l+1-i$ and $\dim(\xi'_{(0,c)})$ over $RP(\varepsilon_{(1,b)})$ is r_j if c occupies the j-th position; this part is $[\bigoplus \xi'_a \to RP(\varepsilon_{(1,b)})_{|L}]$, observing that in this case λ occupies the s-th position. Then the part of $[\bigoplus \xi_a]$ that belongs to the summand

$$\mathcal{N}_l(BO(j_1) \times ... \times BO(j_{2^{k-1}}) \times BO(r_1) \times ... \times BO(r_{s-1}) \\ \times BO(r_s+1) \times BO(r_{s+1}) \times ... \times BO(r_{2^{k-1}-1}))$$

is exactly $[\bigoplus \xi_a \to RP(\varepsilon_{(1,b)})_{|L}]$, where $\xi_{(0,b)} = \xi'_{(0,b)} \oplus \lambda$ and $\xi_a = \xi'_a$ for $a \neq (0,b)$. Now a Whitney number of $\bigoplus \xi'_a \to RP(\varepsilon_{(1,b)})_{|L}$ has the form

$$c^t W_\omega(RP(\varepsilon_{(1,b)})_{|L}) V_\rho(\xi'_{(0,b)}) K[RP(\varepsilon_{(1,b)})_{|L}],$$

where K is a product of classes of the $\xi'_{(0,c)}$ with $c \neq b$. By Fact b) one has that this number is equal to

$$c^t W_\omega(RP(\varepsilon_{(1,b)})_{|L}) P_\rho(c, V(\xi_{(0,b)})) K[RP(\varepsilon_{(1,b)})_{|L}],$$

which is zero because it is a Whitney number of the part of $[\bigoplus \xi_a]$ in question.

This shows that $[\bigoplus \xi'_a] = 0$, and we conclude that

$$0 = F'_k F'_{k-1} ... F'_2 [(RP(\eta), T'_2, ..., T'_k; \lambda, T^*_2, ..., T^*_k)] = F'_k F'_{k-1} ... F'_2 \rho' S'(\gamma'),$$

hence that $S'(\gamma') = 0$. It follows that $\gamma' = F'_1[(W^{n-1}, \Phi')]$, and so

$$\beta' = F'_k F'_{k-1} ... F'_1 [(W^{n-1}, \Phi')];$$

that is, (W^{n-1}, Φ') is a $(Z_2)^k$-action with fixed data bordant to the Whitney sum obtained from $\bigoplus \varepsilon_a \to F$ by omitting the trivial 1-dimensional factor, and the argument is completed.

Remarks. 1) When considered for $k = 2$, the above method provides a proof shorter and clearer than that presented in [5].

2) A consequence of the above proof is that there is a Smith homomorphism $\triangle : \mathcal{K} \to \widehat{\mathcal{N}}_{m-1}((Z_2)^k, 0)$, where $\mathcal{K}$ is the submodule $SI_*(\bigoplus_{r+s=m} \mathcal{N}_{r,s}((Z_2)^{k-1}, 1)) \subset \widehat{\mathcal{N}}_m((Z_2)^k, 0)$. In fact, given $\alpha \in \mathcal{K}$, we choose a representative of α of the form $\sum [(S(\xi \oplus R), A, T'_1, T'_2, .., T'_{k-1})]$, where each $\xi \to V$ is a bundle with $(Z_2)^{k-1}$-action $(T_1, T_2, .., T_{k-1})$, A means the antipodal and $T'_i(v, r) = (T_i(v), r)$, $1 \le i \le k-1$, and we define $\triangle(\alpha) = \sum [(S(\xi), A, T_1, T_2, .., T_{k-1})]$. The above proof shows that this definition does not depend on the particular choice of a representative of the above form. However we are not able to define $\triangle$ on all of $\widehat{\mathcal{N}}_m((Z_2)^k, 0)$ (as in the $k = 1$ case) since equivariant transversality is not valid.

The above remark was pointed out to me by the referee.

4. $(Z_2)^k$-actions fixing $RP(2n) \cup \{p\}$

In this section we will obtain, up to bordism, all possible $(Z_2)^k$-actions fixing $RP(2n) \cup \{p\}$. This will be achieved by putting together the information of the previous section and the methods of [7]. We need first to summarize the main facts of [7]. From a given involution (W, T) we can construct a special family of $(Z_2)^k$-actions, described as follows: for $1 \le t \le k$, let $(Z_2)^k$ act on $W^{2^{t-1}}$, the cartesian product of 2^{t-1} copies of W, by $T_1(x_1, x_2, ..., x_{2^{t-1}}) = (T(x_1), T(x_2), ..., T(x_{2^{t-1}}))$, letting $T_2, ..., T_t$ act by permuting factors so that the points fixed by $T_2, ..., T_t$ form the diagonal copy of W, and letting $T_{t+1}, ..., T_k$ act trivially. We denote this action by $\Gamma^k_t(W, T)$, and we notice that if $\eta \to F$ denotes the fixed data of (W, T) then the fixed data of $\Gamma^k_t(W, T)$ contains 2^{t-1} copies of η, $2^{t-1} - 1$ copies of $\tau(F)$ and $2^k - 2^t$ copies of O; here $\tau(F)$ and O denote, respectively, the tangent bundle and the 0-dimensional bundle over F. More precisely, and taking into account the order of these bundles, this fixed data (with respect to the order of the representations of $(Z_2)^k$ described in Section 2) can be described by using induction on t: it is $\bigoplus_{i=1}^{2^k-1} \varepsilon_{a_i} \to F$, where both $\bigoplus_{i=1}^{2^{t-1}-1} \varepsilon_{a_i} \to F$ and $\bigoplus_{i=2^{t-1}}^{2^t-2} \varepsilon_{a_i}$ are equal to the fixed data of $\Gamma^{t-1}_{t-1}(W, T)$, and where $\varepsilon_{a_1} = \eta$, $\varepsilon_{a_{2^t-1}} = \tau(F)$ for $t > 1$ and $\varepsilon_{a_j} = O$ for $2^t \le j \le 2^k - 1$.

Consider now a fixed smooth closed connected n-dimensional manifold V^n, and as remarked in Section 1 denote by $\mathcal{A}$ the collection of all equivariant bordism classes of involutions containing a representative (W, T) with W connected and $V^n \cup \{p\}$ as fixed point set. Setting $\mathcal{A} = \{[W_i^{n_i}, T_i]\}$, where $n_i = \dim(W_i^{n_i})$, let $\eta_i \to V^n$ denote the normal bundle of V^n in each $W_i^{n_i}$. Evidently, the component of the fixed data of $(W_i^{n_i}, T_i)$ over the point p is the trivial n_i-plane bundle, $R^{n_i} \to p$.

The results of [7] (Theorem 1, page 72, and Theorem 2, page 73), which show that both the order and each individual bundle of the fixed data of a $(Z_2)^k$-action fixing $V^n \cup \{p\}$ are determined, up to bordism, by the collection $\mathcal{A}$ and the operations Γ_t^k, can be summarized as follows: let (M, Φ) be a $(Z_2)^k$-action with fixed data $(\bigoplus_a \varepsilon_a \to V^n) \cup (\bigoplus_a \mu_a \to p)$. Then each ε_a is either bordant to some η_i (and in this case the corresponding μ_a is $R^{n_i} \to p$), or bordant to $\tau(V^n)$ (in this case the corresponding μ_a is O) or equal to O (in this case $\mu_a = O$). Moreover, there are $1 \le t \le k$ and $\sigma \in \mathrm{Aut}((Z_2)^k)$ such that the number of bundles bordant to $\eta_{i's}$ is 2^{t-1} and the number of bundles bordant to $\tau(V^n)$ is $2^{t-1}-1$ (which implies that there are $2^k - 2^t$ remaining zero bundles), and these bundles are included in $\bigoplus \varepsilon_a$ as the corresponding bundles are included in the fixed data of an action of type $\sigma\Gamma_t^k(W, T)$.

In other words, there is a similarity between the fixed data of $\sigma\Gamma_t^k(W, T)$ and (M, Φ), which can be seen through the correspondence between bundles η and bundles bordant to η_i's, between bundles $\tau(F)$ and bundles bordant to $\tau(V^n)$, and between 0-dimensional bundles. Hence, we can see that if, in particular, $\mathcal{A}$ has a single element, then the fixed data of (M, Φ) is very near to the fixed data of $\sigma\Gamma_t^k(W, T)$, where (W, T) is the only element of $\mathcal{A}$. However, we remark that even in this case we cannot guarantee that (M, Φ) is bordant to $\sigma\Gamma_t^k(W, T)$, since the bundles ε_a are determined up to bordism only individually, while the fact just asserted requires a simultaneous bordism to be true.

When $\mathcal{A}$ has more than one element the situation may be much more complex, since in this case there is the possibility of different bordism classes of η_i's occurring in the same fixed data. As it will be seen, this is what happens when $V = RP(2n)$. The collection $\mathcal{A}$ relative to this case was determined by Royster in [2]; next we focus our attention on this collection. Consider the standard endomorphism $\Gamma : \mathcal{N}_*^{Z_2} \to \mathcal{N}_*^{Z_2}$ of degree one (see, for example, [3]) and the augmentation $\varepsilon : \mathcal{N}_*^{Z_2} \to \mathcal{N}_*$; also consider the involution $(RP(2n+1), \tau)$, where

$$\tau[x_0, x_1, ..., x_{2n+1}] = [-x_0, x_1, ..., x_{2n+1}].$$

The fixed data of this involution is

$$(\lambda \to RP(2n)) \cup (R^{2n+1} \to p),$$

where λ is the canonical line bundle. Since $RP(2n) \cup \{p\}$ does not bound, it follows from the strengthened Boardman 5/2-theorem of [1] that there exists $k_n \in Z^+$ such that $\varepsilon\Gamma^{k_n}[(RP(2n+1), \tau)] \neq 0$ and $\varepsilon\Gamma^i[(RP(2n+1), \tau)] = 0$ for all $0 \le i < k_n$. Then for each $0 \le i \le k_n$ the fixed data of $\Gamma^i(RP(2n+1), \tau)$ can be considered, with no loss, as being $(\lambda \oplus R^i \to RP(2n)) \cup (R^{2n+i+1} \to p)$. Royster proved in [2] that the collection $\mathcal{A}$ in question is $\mathcal{A} = \{\Gamma^i[(RP(2n+1), \tau)],\ 0 \le i \le k_n\}$.

We now proceed to determine the $(Z_2)^k$-actions fixing $RP(2n) \cup \{p\}$. Suppose then that (M, Φ) is a $(Z_2)^k$-action fixing $RP(2n) \cup \{p\}$, and let $(\bigoplus \varepsilon_a \to RP(2n)) \cup (\bigoplus \mu_a \to p)$ denote the fixed data. By the above facts, there is $1 \le t \le k$ such that:

i) There are 2^{t-1} bundles $\varepsilon_{a_1}, \varepsilon_{a_2}, ..., \varepsilon_{a_{2^{t-1}}}$ bordant, respectively, to $\lambda \oplus R^{i_1}, \lambda \oplus R^{i_2}, ..., \lambda \oplus R^{i_{2^{t-1}}}$, where $0 \le i_1, i_2, ..., i_{2^{t-1}} \le k_n$; for each a_l, $1 \le l \le 2^{t-1}$, the corresponding μ_{a_l} is $R^{2n+i_l+1} \to p$.

ii) There are $2^{t-1}-1$ bundles $\varepsilon_{b_1}, \varepsilon_{b_2}, ..., \varepsilon_{b_{2^{t-1}-1}}$ bordant to $\tau(RP(2n))$, and for each b_l, $1 \le l \le 2^{t-1}-1$, the corresponding μ_{b_l} is the zero bundle.

iii) The remaining $2^k - 2^t$ ε_a's and μ_a's are the zero bundle.

iv) The order with respect to which the above bundles are included in $\bigoplus \varepsilon_a$ and $\bigoplus \mu_a$ is as described above for some $\sigma \in \mathrm{Aut}((Z_2)^k)$.

We assume that $a_1, a_2, ..., a_{2^{t-1}}$ follow the order with respect to which the bundles ε_{a_l} are included in $\bigoplus \varepsilon_a$. Now a bundle $\eta \to RP(2n)$ bordant to $\lambda \oplus R^i$ necessarily has $w_1(\eta) = \alpha$ and $w_r(\eta) = 0$ for $r > 1$, where $\alpha \in H^1(RP(2n), Z_2)$ is the generator; on the other hand, one knows from [11] that if η is bordant to $\tau(RP(2n))$, then $W(\eta) = (1+\alpha)^{2n+1}$. These facts produce the desired simultaneous bordism mentioned before; that is, they imply that $(\bigoplus \varepsilon_a) \cup (\bigoplus \mu_a)$ is bordant, as an element of $\bigoplus \mathcal{N}_p(BO(n_1) \times ... \times BO(n_{2^k-1}))$, to the Whitney sum obtained by replacing each ε_{a_l} by $\lambda \oplus R^{i_l}$ and each ε_{b_l} by $\tau(RP(2n))$. To complete the classification, all that remains is to exhibit a $(Z_2)^k$-action with this latter Whitney sum as fixed data. To do this, consider $x = \max\{i_1, i_2, ..., i_{2^{t-1}}\}$, and set $e_l = x - i_l$, $1 \leq l \leq 2^{t-1}$. By applying the theorem of Section 3 e_1 times to the fixed data of $\sigma\Gamma_t^k\Gamma^x(RP(2n+1), \tau)$ one obtains a $(Z_2)^k$-action (M_1, Φ_1) whose fixed data is obtained from the fixed data of $\sigma\Gamma_t^k\Gamma^x(RP(2n+1), \tau)$ by replacing the first $\lambda \oplus R^x$ by $\lambda \oplus R^{i_1}$. Next, we apply the theorem e_2 times to the fixed data of (M_1, Φ_1) to obtain a $(Z_2)^k$-action (M_2, Φ_2) with fixed data obtained from the fixed data of (M_1, Φ_1) by replacing the second $\lambda \oplus R^x$ by $\lambda \oplus R^{i_2}$; and so on; we end by obtaining a $(Z_2)^k$-action $(M_{2^{t-1}}, \Phi_{2^{t-1}})$ with the desired fixed data (since a $(Z_2)^k$-action cannot fix precisely one point [3, 31.3], $M_{2^{t-1}}$ is necessarily connected; such an action can be explicitly obtained by using the method of [4]).

In other words, for each sequence $\varrho = (\sigma, t; i_1, i_2, ..., i_{2^{t-1}})$, where $\sigma \in \mathrm{Aut}((Z_2)^k)$, $1 \leq t \leq k$ and $0 \leq i_1, i_2, ..., i_{2^{t-1}} \leq k_n$, and for $x = \max\{i_1, i_2, ..., i_{2^{t-1}}\}$, we can construct from $\sigma\Gamma_t^k\Gamma^x(RP(2n+1), \tau)$ a $(Z_2)^k$-action $(M_\varrho, \Phi_\varrho)$ by adopting the above procedure, and what we have proved is that our original action (M, Φ) is necessarily bordant to one of these actions.

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Departamento de Matemática, Universidade Federal de São Carlos, Caixa Postal 676, CEP 13.565-905, São Carlos, SP, Brazil

E-mail address: pergher@dm.ufscar.br